Kuwait University — Math 102 — May 24th, 2000

Dept. of Math. and Comp. Sc. — Final Examination — Duration: 2 hours

Calculators, mobile phones and pagers are NOT ALLOWED

1. Let $f(x) = \ln(1 + \sqrt{x^2 - 1}),\ x \geq 1$. Show that f^{-1} exists and find $f^{-1}(x)$.

3 points

2. Find the limit

$$\lim_{x\to\infty} \left(1 + \frac{1}{2\sqrt{x}}\right)^{\sqrt{x}}$$

3 points

3. Evaluate the following integrals:

(a) $\displaystyle\int \ln(x^2+4)\,dx$ (b) $\displaystyle\int \frac{1}{x+2x^2+x^3}\,dx$

(c) $\displaystyle\int \frac{1}{(x+2)\sqrt{4x+x^2}}\,dx$ (d) $\displaystyle\int \sqrt[3]{\sec x}\,\tan^3 x\,dx.$

4 points each

4 Determine whether the improper integral $\displaystyle\int_0^1 \frac{4}{x(\ln^2 x + 4)}\,dx$ is convergent or divergent, and if convergent find its value.

4 points

5. (a) Sketch the graphs of the polar curves

$$r = 2 + 2\cos\theta, \qquad r = 2\sqrt{3}\sin\theta.$$

(b) Find the area of the region that is outside the graph of $r = 2 + 2\cos\theta$ and inside the graph of $r = 2\sqrt{3}\sin\theta$ (the intersection points of these curves are on the line $\theta = \pi/3$)

2+4 points

6. (a) Find an equation of the plane π that passes through the points

$$Q(1,0,1),\ R(2,1,1),\ S(3,-1,3).$$

(b) Find parametric equations for the line through the point $P(1,3,4)$ that is perpendicular to the plane π in question 6. (a).

(c) Calculate the distance from the point $P(1,3,4)$ to the plane π.

3+3+2 points

Total 40 points

$f(x) = \ln(1+\sqrt{x^2-1}) \quad x \geq 1$

$f'(x) = \frac{1}{(1+\sqrt{x^2-1})} \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}+x^2-1} \geq 0 \ \forall x \geq 1$

$\therefore f$ is increasing on $[1,\infty)$. $\therefore f$ is one-to-one $\therefore f^{-1}$ exists.

$y = \ln(1+\sqrt{x^2-1})$

$e^y = 1+\sqrt{x^2-1} \qquad \sqrt{x^2-1} = e^y - 1$

$x^2 - 1 = (e^y-1)^2$

$x^2 = (e^y-1)^2+1$

$x = \sqrt{(e^y-1)^2+1} \qquad \therefore f^{-1}(x) = \sqrt{(e^x-1)^2+1}$

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2) $\lim_{x\to\infty} \left(1+\frac{1}{2\sqrt{x}}\right)^{\sqrt{x}} \to 1^\infty$

$\ln y = \ln\left(1+\frac{1}{2\sqrt{x}}\right)^{\sqrt{x}} = \sqrt{x}\ \ln\left(1+\frac{1}{2\sqrt{x}}\right)$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \sqrt{x}\ \ln\left(1+\frac{1}{2\sqrt{x}}\right) \to (\infty)(0)$

$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{2\sqrt{x}}\right)}{\frac{1}{\sqrt{x}}} \longrightarrow \frac{0}{0}$, by l'Hôpital's rule.

$= \lim_{x\to\infty} \frac{1}{\cancel{-}\frac{\cancel{\sqrt{x}}}{2}} \cdot \frac{1}{1+\frac{1}{2\sqrt{x}}} \cdot (\cancel{-}\cancel{\sqrt{x}})$

$= \lim_{x\to\infty} \frac{2}{1+\frac{1}{2\sqrt{x}}} = 2$ ■

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3) (a) $\int \ln(x^2+4)\,dx$ 1037@mail.com

$dv = dx \qquad u = \ln(x^2+4)$

$v = x \qquad du = \frac{1}{x^2+4} \cdot 2x \cdot dx$

$= x\ln(x^2+4) - 2\int \frac{x^2+4-4}{x^2+4}dx$

$x\ln(x^2+4) - 2\left[\int dx - 4\int \frac{1}{\ } dx\right]$